

# SEPARATORS OF POINTS IN A MULTIPROJECTIVE SPACE

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**ABSTRACT.** In this note we develop some of the properties of separators of points in a multiprojective space. In particular, we prove multigraded analogs of results of Geramita, Maroscia, and Roberts relating the Hilbert function of  $\mathbb{X}$  and  $\mathbb{X} \setminus \{P\}$  via the degree of a separator, and Abrescia, Bazzotti, and Marino relating the degree of a separator to shifts in the minimal multigraded free resolution of the ideal of points.

## 1. INTRODUCTION

Let  $R = k[x_{1,0}, \dots, x_{1,n_1}, \dots, x_{r,0}, \dots, x_{r,n_r}]$  be the  $\mathbb{N}^r$ -graded polynomial ring with  $\deg x_{i,j} = e_i$ , the  $i$ th standard basis vector in  $\mathbb{N}^r$ , and  $k$  an algebraically closed field of characteristic zero. If  $\mathbb{X} = \{P_1, \dots, P_s\}$  is a finite set of points in a multiprojective space  $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ , then  $R/I_{\mathbb{X}}$  is the associated  $\mathbb{N}^r$ -graded coordinate ring. If  $P \in \mathbb{X}$ , then the multihomogeneous form  $F \in R$  is a **separator for**  $P$  if  $F(P) \neq 0$  and  $F(Q) = 0$  for all  $Q \in \mathbb{X} \setminus \{P\}$ . The **degree of a point**  $P \in \mathbb{X}$  is the set

$$\deg_{\mathbb{X}}(P) = \min\{\deg F \mid F \text{ is a separator for } P \in \mathbb{X}\}.$$

Here, we are using the partial order on  $\mathbb{N}^r$  defined by  $(i_1, \dots, i_r) \succeq (j_1, \dots, j_r)$  whenever  $i_t \geq j_t$  for all  $t = 1, \dots, r$ . The goal of this note is to record some of the properties of a separator of a point and its degree in a multigraded context.

The notion of a separator was first introduced for sets of points  $\mathbb{X}$  in  $\mathbb{P}^n$  by Orecchia [18] to investigate the conductor of  $A = R/I_{\mathbb{X}}$ , that is, the largest ideal  $J$  of  $A$  that corresponds with its extension in the integral closure  $\bar{A}$ . It was shown that the degrees of the minimal generators of  $J$  corresponded to the degrees of the points  $P \in \mathbb{X}$ . As later shown by Geramita, Maroscia, and Roberts [6], the degree of a point  $P$  allows one to relate the Hilbert function of  $\mathbb{X}$  to that of  $\mathbb{X} \setminus \{P\}$ . Abrescia, Bazzotti, and Marino [1] demonstrated that  $\deg_{\mathbb{X}}(P)$  was also linked to the shifts appearing in the minimal free graded resolution of  $R/I_{\mathbb{X}}$ . Further properties of separators in the graded case can be found in [2, 3, 13, 14, 19], among others.

The study of separators of points in a multigraded setting was initiated by Marino [15, 16, 17] who studied separators of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Note that when  $r \geq 2$ , then it may happen that  $|\deg_{\mathbb{X}}(P)| \geq 2$ , thus presenting one of the fundamental differences between the study of separators of points in  $\mathbb{P}^n$  versus those in  $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ . Marino showed that  $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  is arithmetically Cohen-Macaulay (ACM) if and only if for

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every  $P \in \mathbb{X}$ ,  $|\deg_{\mathbb{X}}(P)| = 1$ . More recently, the authors [12] extended some of Marino's results to an arbitrary multiprojective space; in particular, if  $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$  and is ACM, then every point  $P \in \mathbb{X}$  has  $|\deg_{\mathbb{X}}(P)| = 1$ , but the converse no longer holds.

While a cursory introduction to the properties of separators appears in [12], in this paper we wish to provide a more systematic introduction, thereby extending our understanding of points in a multiprojective space (see, for example, [7, 8, 9, 10, 11, 20, 21, 22], for more on these points). In Section 2, we relate the Hilbert functions of  $\mathbb{X}$  and  $\mathbb{X} \setminus \{P\}$  using the set  $\deg_{\mathbb{X}}(P)$  (see Theorem 2.2), thus introducing a multigraded analog of a result of Geramita, Maroscia, and Roberts [6]. The main result (Theorem 3.2) of Section 3 relates  $\deg_{\mathbb{X}}(P)$  to the shifts at the end of the multigraded resolution of  $R/I_{\mathbb{X}}$  when  $\mathbb{X}$  is ACM. This result extends a result of Abrescia, Bazzotti, and Marino [1] first proved for separators of points in  $\mathbb{P}^n$ . In the final section, we restrict to the case of ACM points in  $\mathbb{P}^1 \times \mathbb{P}^1$  and their separators. In particular, we show (see Theorems 4.6 and 4.9) that the converse of Theorem 3.2 holds in  $\mathbb{P}^1 \times \mathbb{P}^1$ .

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## 2. SEPARATORS, HILBERT FUNCTIONS, AND ACMNESS

We continue to use the notation from the introduction. If  $S \subseteq \mathbb{N}^r$ , then  $\min S$  denotes the set of the minimal elements of  $S$  with respect to the partial ordering  $\succeq$ . For any  $\underline{i} \in \mathbb{N}^r$ , define  $D_{\underline{i}} := \{\underline{j} \in \mathbb{N}^r \mid \underline{j} \succeq \underline{i}\}$ . For any finite set  $S = \{\underline{s}_1, \dots, \underline{s}_p\} \subseteq \mathbb{N}^r$ , we set

$$D_S := \bigcup_{\underline{s} \in S} D_{\underline{s}}.$$

Note that  $\min D_S = S$ ; thus  $D_S$  can be viewed as the largest subset of  $\mathbb{N}^r$  whose minimal elements are the elements of  $S$ .

Let  $\mathbb{X}$  be a set of distinct points in  $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$  and  $P \in \mathbb{X}$ . We say that the multihomogeneous form  $F \in R$  is a **minimal separator for  $P$**  if  $F$  is a separator for  $P$ , and if there does not exist a separator  $G$  for  $P$  with  $\deg G \prec \deg F$ . Note that

$$\deg_{\mathbb{X}}(P) = \{\deg F \mid F \text{ is a minimal separator of } P \in \mathbb{X}\}.$$

**Lemma 2.1.** *Let  $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$  be a set of points and let  $P \in \mathbb{X}$ . Then for every  $\underline{i} \in D_{\deg_{\mathbb{X}}(P)}$  there exists a form  $F$  with  $\deg F = \underline{i}$  that is a separator of  $P$ .*

*Proof.* Fix a  $P \in \mathbb{X}$ . For each  $i = 1, \dots, r$ , there exists a form  $L_i$  with  $\deg L_i = e_i$  such that  $L_i(P) \neq 0$ . Geometrically picking  $L_i$  corresponds to picking a hyperplane in  $\mathbb{P}^{n_i}$  that misses the  $i$ th coordinates of the points of  $\mathbb{X}$ . For any  $\underline{i} \in D_{\deg_{\mathbb{X}}(P)}$ , there exists  $\underline{\alpha} \in \deg_{\mathbb{X}}(P)$  with  $\underline{i} \succeq \underline{\alpha}$ . Let  $F'$  be a minimal separator of  $P$  with  $\deg F' = \underline{\alpha}$ . Then the desired separator is

$$F = F' \prod_{j=1}^r L_j^{i_j - \alpha_j}$$

where  $\underline{i} = (i_1, \dots, i_r)$  and  $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ . □

If  $I$  is a multihomogeneous ideal of  $R$ , then the **Hilbert function of  $S = R/I$**  is the numerical function  $H_S : \mathbb{N}^r \rightarrow \mathbb{N}$  defined by

$$H_S(\underline{i}) := \dim_k S_{\underline{i}} = \dim_k R_{\underline{i}} - \dim_k (I)_{\underline{i}} \text{ for all } \underline{i} \in \mathbb{N}^r.$$

When  $S = R/I_{\mathbb{X}}$  is the coordinate ring of a set of points  $\mathbb{X}$ , then we usually say  $H_S$  is the **Hilbert function of  $\mathbb{X}$** , and write  $H_{\mathbb{X}}$ .

If  $P \in \mathbb{X}$  and  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$ , then  $H_{\mathbb{Y}}$  can be computed from  $H_{\mathbb{X}}$  and  $\deg_{\mathbb{X}}(P)$  as demonstrated below. We view this result as a multigraded version of [6, Lemma 2.3].

**Theorem 2.2.** *Let  $\mathbb{X}$  be a set of distinct points in  $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ , and let  $P \in \mathbb{X}$  be any point. If  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$ , then*

$$H_{\mathbb{Y}}(\underline{i}) = \begin{cases} H_{\mathbb{X}}(\underline{i}) & \text{if } \underline{i} \notin D_{\deg_{\mathbb{X}}(P)} \\ H_{\mathbb{X}}(\underline{i}) - 1 & \text{if } \underline{i} \in D_{\deg_{\mathbb{X}}(P)}. \end{cases}$$

*Proof.* It was shown in [12, Theorem 5.3] that there exists a finite set  $S \subseteq \mathbb{N}^r$  such that

$$H_{\mathbb{Y}}(\underline{i}) = \begin{cases} H_{\mathbb{X}}(\underline{i}) & \text{if } \underline{i} \notin D_S \\ H_{\mathbb{X}}(\underline{i}) - 1 & \text{if } \underline{i} \in D_S. \end{cases}$$

It therefore suffices to show that  $S = D_{\deg_{\mathbb{X}}(P)}$ . Suppose  $\underline{i} \notin D_{\deg_{\mathbb{X}}(P)}$  but  $H_{\mathbb{Y}}(\underline{i}) = H_{\mathbb{X}}(\underline{i}) - 1$ . This implies that  $\dim_k (I_{\mathbb{Y}})_{\underline{i}} = \dim_k (I_{\mathbb{X}})_{\underline{i}} + 1$ , or equivalently, there exists a form  $F \in (I_{\mathbb{Y}})_{\underline{i}} \setminus (I_{\mathbb{X}})_{\underline{i}}$ . But then  $F$  vanishes at all the points of  $\mathbb{Y}$  but not at all the points of  $\mathbb{X}$ , i.e.,  $F$  does not vanish at  $P$ . So  $F$  is a separator of  $P$ , and thus there exists an  $\underline{\alpha} \in \deg_{\mathbb{X}}(P)$  such that  $\underline{i} \succeq \underline{\alpha}$ . But this contradicts the fact that  $\underline{i} \notin D_{\deg_{\mathbb{X}}(P)}$ . So  $H_{\mathbb{Y}}(\underline{i}) = H_{\mathbb{X}}(\underline{i})$ .

Now suppose that  $\underline{i} \in D_{\deg_{\mathbb{X}}(P)}$  but  $H_{\mathbb{Y}}(\underline{i}) = H_{\mathbb{X}}(\underline{i})$ . By Lemma 2.1 there exists a form  $F$  with  $\deg F = \underline{i}$  such that  $F$  is a separator of  $P$ . So  $F \in (I_{\mathbb{Y}})_{\underline{i}}$  but  $F \notin (I_{\mathbb{X}})_{\underline{i}}$ . This contradicts the fact that  $H_{\mathbb{Y}}(\underline{i}) = H_{\mathbb{X}}(\underline{i})$  implies  $\dim_k (I_{\mathbb{Y}})_{\underline{i}} = \dim_k (I_{\mathbb{X}})_{\underline{i}}$ . So  $H_{\mathbb{Y}}(\underline{i}) = H_{\mathbb{X}}(\underline{i}) - 1$ .  $\square$

**Remark 2.3.** Theorem 2.2 shows that  $\deg_{\mathbb{X}}(P) = \min\{\underline{i} \in \mathbb{N}^r \mid H_{\mathbb{X}}(\underline{i}) \neq H_{\mathbb{Y}}(\underline{i})\}$ . One can therefore compute  $\deg_{\mathbb{X}}(P)$  by comparing the Hilbert functions of  $\mathbb{X}$  and  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$ .

If  $\mathbb{X} \subseteq \mathbb{P}^n$ , then  $\mathbb{N}$  is a totally ordered set, so we can study *the* degree of a point  $P \in \mathbb{X}$  (as in [1, 2, 3, 13, 14, 18, 19]). In the multigraded case the set  $\deg_{\mathbb{X}}(P) = \{\underline{\alpha}_1, \dots, \underline{\alpha}_s\} \subseteq \mathbb{N}^r$  may have  $s \geq 1$ . However, if  $F$  is a minimal separator of  $P$  with  $\deg F = \underline{\alpha}_i \in \deg_{\mathbb{X}}(P)$ , then the equivalence class of  $F$  in  $R/I_{\mathbb{X}}$  is unique up to scalar multiplication.

**Theorem 2.4** ([12, Corollary 5.4]). *Suppose  $\deg_{\mathbb{X}}(P) = \{\underline{\alpha}_1, \dots, \underline{\alpha}_s\} \subseteq \mathbb{N}^r$ . If  $F$  and  $G$  are any two minimal separators of  $P$  with  $\deg F = \deg G = \underline{\alpha}_i$ , then there exists  $0 \neq c \in k$  such that  $\overline{G} = c\overline{F} \in R/I_{\mathbb{X}}$ .*

If  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  for some  $P \in \mathbb{X}$ , then the defining ideals of  $I_P$ ,  $I_{\mathbb{Y}}$  and  $I_{\mathbb{X}}$  are related via the separators of  $P$ , as demonstrated below.

**Theorem 2.5.** *Let  $\mathbb{X}$  be a set of points in  $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ ,  $P \in \mathbb{X}$ , and  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$ .*

- (i) *If  $F$  is a separator of a point  $P$ , then  $(I_{\mathbb{X}} : F) = I_P$ .*
- (ii) *If  $\deg_{\mathbb{X}}(P) = \{\underline{\alpha}_1, \dots, \underline{\alpha}_s\}$ , and if  $F_i$  is a minimal separator of  $P$  with  $\deg F_i = \underline{\alpha}_i$ , then  $I_{\mathbb{Y}} = (I_{\mathbb{X}}, F_1, \dots, F_s)$ .*

*Proof.* Statement (i) is [12, Theorem 5.5]. For (ii), the containment  $(I_{\mathbb{X}}, F_1, \dots, F_s) \subseteq I_{\mathbb{Y}}$  is clear since  $F_i \in I_{\mathbb{Y}}$  for each  $i$  and  $I_{\mathbb{X}} \subseteq I_{\mathbb{Y}}$ . Now, if  $\underline{i} \notin D_{\deg_{\mathbb{X}}(P)}$ , then  $(I_{\mathbb{X}}, F_1, \dots, F_s)_{\underline{i}} = (I_{\mathbb{X}})_{\underline{i}} = (I_{\mathbb{Y}})_{\underline{i}}$  where the last equality is a consequence of Theorem 2.2. On the other hand, if  $\underline{i} \in D_{\deg_{\mathbb{X}}(P)}$ , then

$$\dim_k(I_{\mathbb{X}})_{\underline{i}} < \dim_k(I_{\mathbb{X}}, F_1, \dots, F_s)_{\underline{i}} \leq \dim_k(I_{\mathbb{Y}})_{\underline{i}} \leq \dim_k(I_{\mathbb{X}})_{\underline{i}} + 1.$$

The last inequality follows from Theorem 2.2. We are forced to have  $\dim_k(I_{\mathbb{X}})_{\underline{i}} + 1 = \dim_k(I_{\mathbb{X}}, F_1, \dots, F_s)_{\underline{i}} = \dim_k(I_{\mathbb{Y}})_{\underline{i}}$  i.e.,  $(I_{\mathbb{X}}, F_1, \dots, F_s)_{\underline{i}} = (I_{\mathbb{Y}})_{\underline{i}}$ . Since  $(I_{\mathbb{X}}, F_1, \dots, F_s) \subseteq I_{\mathbb{Y}}$ , and  $(I_{\mathbb{X}}, F_1, \dots, F_s)_{\underline{i}} = (I_{\mathbb{Y}})_{\underline{i}}$  for all  $\underline{i} \in \mathbb{N}^r$ , this completes the proof.  $\square$

We end this section discussing the connection between separators and the ACMness of a set of points. For any finite set of points  $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ , it can be shown (see, for example [12, Theorem 2.1]) that  $\dim R/I_{\mathbb{X}} = r$  and  $1 \leq \text{depth } R/I_{\mathbb{X}} \leq r$ . When  $\text{depth } R/I_{\mathbb{X}} = r$ , then we say  $\mathbb{X}$  is **arithmetically Cohen-Macaulay** (ACM). Although it remains an open problem to classify ACM sets of points in a multiprojective space (see [12] for some work on this problem), it can be shown that the separators of ACM sets of points have a particularly nice property:

**Theorem 2.6** ([12, Theorem 5.7]). *Let  $\mathbb{X}$  be any ACM set of points in  $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ . Then  $|\deg_{\mathbb{X}}(P)| = 1$  for every  $P \in \mathbb{X}$ .*

In the case of ACM sets of points in  $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ , we can talk about *the* degree of a point, and in this case we usually abuse notation and write  $\deg_{\mathbb{X}}(P) = \alpha$  instead of  $\deg_{\mathbb{X}}(P) = \{\alpha\}$ . Although the converse of Theorem 2.6 fails to hold in general (see [12, Example 5.10]), the converse holds in  $\mathbb{P}^1 \times \mathbb{P}^1$  as first demonstrated by Marino:

**Theorem 2.7** ([17]). *Let  $\mathbb{X}$  be a finite set of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Then  $\mathbb{X}$  is ACM if and only if  $|\deg_{\mathbb{X}}(P)| = 1$  for every  $P \in \mathbb{X}$ .*

### 3. THE DEGREE OF A POINT AND THE MINIMAL RESOLUTION

By Theorem 2.2, if we can compute  $\deg_{\mathbb{X}}(P)$ , then  $H_{\mathbb{Y}}$  can be computed from  $H_{\mathbb{X}}$  where  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$ . It is therefore of interest to identify what finite subsets  $S \subseteq \mathbb{N}^r$  can be the degree of a point. In this section, we show that under the extra hypothesis that  $\mathbb{X}$  is ACM, information about  $\deg_{\mathbb{X}}(P)$  can be read from the last shift in the minimal multigraded resolution of  $I_{\mathbb{X}}$ . Our result can be seen as a multigraded analog of a theorem of Abrescia, Bazzotti, and Marino [1]. We begin with a lemma.

**Lemma 3.1.** *Let  $P \in \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$  be any point. Then the minimal  $\mathbb{N}^r$ -graded free resolution of  $R/I_P$  has the form*

$$0 \rightarrow \mathbb{G}_t \rightarrow \mathbb{G}_{t-1} \rightarrow \dots \rightarrow \mathbb{G}_1 \rightarrow R \rightarrow R/I_P \rightarrow 0$$

where  $t = \sum_{i=1}^r n_i$  and  $\mathbb{G}_t = R(-n_1, -n_2, \dots, -n_r)$ .

*Proof.* Because  $I_P$  is a complete intersection, the conclusions follow from the Koszul resolution, taking into account the multigrading.  $\square$

**Theorem 3.2.** *Let  $\mathbb{X}$  be a finite set of points in  $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ , and furthermore, suppose that  $\mathbb{X}$  is ACM. Let  $P \in \mathbb{X}$ , and suppose that  $\deg_{\mathbb{X}}(P) = \underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ . Let*

$$0 \rightarrow \mathbb{F}_t = \bigoplus_{\underline{i} \in S_t} R(-\underline{i}) \rightarrow \cdots \rightarrow \mathbb{F}_1 \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0$$

*be the minimal  $\mathbb{N}^r$ -graded free resolution of  $R/I_{\mathbb{X}}$  where  $t = \sum_{i=1}^r n_i$ .*

- (i) *If  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  is ACM, then  $(\alpha_1 + n_1, \dots, \alpha_r + n_r) \in S_t$ , that is,  $\deg_{\mathbb{X}}(P) + (n_1, \dots, n_r)$  appears as a shift in the last free  $R$ -module.*
- (ii) *If  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  is not ACM, then  $\text{depth}(R/I_{\mathbb{Y}}) = r - 1$ .*

*Proof.* Because  $\mathbb{X}$  is ACM, by Theorem 2.6  $\deg_{\mathbb{X}}(P) = \alpha$  for some  $\alpha \in \mathbb{N}^r$ . Let  $F$  be any minimal separator of  $P$ . Hence  $\deg F = \alpha$ , and by Theorem 2.5,  $I_P = (I_{\mathbb{X}} : F)$  and  $(I_{\mathbb{X}}, F) = I_{\mathbb{Y}}$ . We then have the short exact sequence

$$(3.1) \quad 0 \rightarrow R/(I_{\mathbb{X}} : F)(-\alpha) = R/I_P(-\alpha) \xrightarrow{\times \bar{F}} R/I_{\mathbb{X}} \rightarrow R/(I_{\mathbb{X}}, F) = R/I_{\mathbb{Y}} \rightarrow 0.$$

By Lemma 3.1, the resolution of  $R/I_P$  has form

$$0 \rightarrow R(-n_1, \dots, -n_r) \rightarrow \mathbb{G}_{t-1} \rightarrow \cdots \rightarrow \mathbb{G}_1 \rightarrow R \rightarrow R/I_P \rightarrow 0.$$

Applying the mapping cone construction to (3.1) we get a resolution of  $R/I_{\mathbb{Y}}$ :

$$(3.2) \quad \mathcal{H} : 0 \rightarrow R(-\alpha_1 - n_1, \dots, -\alpha_r - n_r) \rightarrow \mathbb{F}_t \oplus \mathbb{G}_{t-1}(-\underline{\alpha}) \rightarrow \cdots \\ \rightarrow \mathbb{F}_2 \oplus \mathbb{G}_1(-\underline{\alpha}) \rightarrow \mathbb{F}_1 \oplus R(-\underline{\alpha}) \rightarrow R \rightarrow R/I_{\mathbb{Y}} \rightarrow 0.$$

If  $\mathbb{Y}$  is ACM, then the above resolution cannot be minimal because it is too long. So  $\mathcal{H} = \mathcal{F} \oplus \mathcal{G}$  where  $\mathcal{F}$  is the minimal resolution of  $R/I_{\mathbb{Y}}$  and  $\mathcal{G}$  is isomorphic to the trivial complex (see [5, Theorem 20.2]). In particular  $R(-\alpha_1 - n_1, \dots, -\alpha_r - n_r)$  must be part of the trivial complex  $\mathcal{G}$ , and thus, to obtain a minimal resolution,  $R(-\alpha_1 - n_1, \dots, -\alpha_r - n_r)$  must cancel with something in  $\mathbb{F}_t \oplus \mathbb{G}_{t-1}(-\underline{\alpha})$ . Since  $R(-\alpha_1 - n_1, \dots, -\alpha_r - n_r)$  does not appear in  $\mathbb{G}_{t-1}(-\underline{\alpha})$ , there exists a shift  $\underline{i} \in S_t$  such that  $\underline{i} = (\alpha_1 + n_1, \dots, \alpha_r + n_r)$ , thus proving (i).

For (ii) the mapping cone resolution gives a (not necessarily minimal) resolution of  $R/I_{\mathbb{Y}}$  that cannot be shortened, because otherwise  $\mathbb{Y}$  would be ACM. So,  $R/I_{\mathbb{Y}}$  has projective dimension  $t + 1$ . Now apply the Auslander-Buchsbaum formula.  $\square$

We can use the above result to show that some sets of points are not ACM.

**Corollary 3.3.** *Let  $\mathbb{X}$  be an ACM scheme in  $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ , and  $P \in \mathbb{X}$ . If  $\deg_{\mathbb{X}}(P) + (n_1, \dots, n_r)$  is not a shift of the last syzygy module of  $R/I_{\mathbb{X}}$ , then  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  is not ACM.*

**Example 3.4.** The converse of Theorem 3.2 (i) is false in general (we will show it is true in  $\mathbb{P}^1 \times \mathbb{P}^1$  in the next section). Let  $P_1, \dots, P_6$  be six points in general position (that is, no more than two points on a line, and no five points on a conic) in  $\mathbb{P}^2$ , and set  $Q_{i,j} := P_i \times P_j \in \mathbb{P}^2 \times \mathbb{P}^2$ . Consider the following set of 28 points:

$$\mathbb{X} = \{Q_{1,1}, Q_{1,2}, Q_{1,3}, Q_{1,4}, Q_{1,5}, Q_{1,6}, Q_{2,1}, Q_{2,2}, Q_{2,3}, Q_{2,4}, Q_{2,6}, Q_{3,1}, Q_{3,2}, Q_{3,5}, Q_{3,6}, \\ Q_{4,1}, Q_{4,2}, Q_{4,5}, Q_{4,6}, Q_{5,1}, Q_{5,3}, Q_{5,6}, Q_{6,1}, Q_{6,2}, Q_{6,3}, Q_{6,4}, Q_{6,5}, Q_{6,6}\}.$$

Then  $\mathbb{X}$  is ACM, since the minimal bigraded resolution has form

$$0 \rightarrow R(-3, -4)^4 \oplus R(-4, -3)^4 \oplus R(-4, -4) \rightarrow R^{32} \rightarrow R^{38} \rightarrow R^{16} \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0$$

where we have suppressed all the other bigraded shifts.

We remove the point  $Q_{2,2}$  to form the set  $\mathbb{Y} = \mathbb{X} \setminus \{Q_{2,2}\}$ . By comparing the Hilbert functions of  $\mathbb{Y}$  and  $\mathbb{X}$  (see Remark 2.3), we find that  $\deg_{\mathbb{X}}(Q_{2,2}) = (2, 2)$ . Now  $\deg_{\mathbb{X}}(Q_{2,2}) + (2, 2) = (4, 4)$  is a shift that appears in the minimal multigraded resolution of  $I_{\mathbb{X}}$ . However,  $\mathbb{Y}$  is not ACM because  $\mathbb{Y}$  is the nonACM set of points of [12, Example 3.3].

**Example 3.5.** In the proof of Theorem 3.2, we saw that (3.2) gives a resolution of  $I_{\mathbb{Y}}$ . When  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  is ACM, this resolution is not minimal because the resolution can be shortened. However, even when we shorten the resolution by cancelling out  $R(-\alpha - n_1, \dots, -\alpha_r - n_r)$  with a term in  $\mathbb{F}_{t-1}$ , the resulting resolution may still not be minimal.

For example, let  $P_{i,j} := [1 : i] \times [1 : j] \in \mathbb{P}^1 \times \mathbb{P}^1$ , and consider the set

$$\mathbb{X} = \{P_{1,1}, P_{1,2}, P_{1,3}, P_{1,4}, P_{1,5}, P_{2,1}, P_{2,2}, P_{2,3}, P_{2,4}, P_{3,1}, P_{3,2}, P_{3,3}, P_{3,4}, P_{4,1}, P_{4,2}, P_{4,3}, P_{5,1}, P_{5,2}, P_{6,1}\}.$$

Set  $\mathbb{Y} = \mathbb{X} \setminus \{P_{3,4}\}$ . We have that  $\deg_{\mathbb{X}}(P_{3,4}) = (2, 3)$ ; in fact, a minimal separator is  $F = L_1 L_2 R_1 R_2 R_3$  where  $L_i = ix_0 - x_1$  is the degree  $(1, 0)$  form that passes through  $[1 : i]$  and  $R_j = jy_0 - y_1$  is the degree  $(0, 1)$  form that passes through  $[1 : j]$  in  $R = k[x_0, x_1, y_0, y_1]$ . The resolution of  $R/I_{\mathbb{X}}$  is:

$$\begin{aligned} 0 \rightarrow R(-1, -5) \oplus R(-3, -4) \oplus R(-4, -3) \oplus R(-5, -2) \oplus R(-6, -1) \rightarrow \\ \rightarrow R(0, -5) \oplus R(-1, -4) \oplus R(-3, -3) \oplus R(-4, -2) \oplus R(-5, -1) \oplus R(-6, 0) \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0. \end{aligned}$$

The mapping cone construction gives the resolution:

$$\begin{aligned} 0 \rightarrow R(-3, -4) \rightarrow R(-3, -4) \oplus R(-1, -5) \oplus R(-4, -3) \oplus R(-5, -2) \oplus R(-6, -1) \oplus R(-2, -4) \oplus R(-3, -3) \\ \rightarrow R(0, -5) \oplus R(-1, -4) \oplus R(-4, -2) \oplus R(-5, -1) \oplus R(-6, 0) \oplus R(-3, -3) \oplus R(-2, -3) \\ \rightarrow R \rightarrow R/I_{\mathbb{Y}} \rightarrow 0. \end{aligned}$$

Since  $\mathbb{Y}$  is ACM, the terms  $R(-3, -4)$  at the last and second last step cancel out. However, the remaining resolution is not a minimal resolution because the minimal resolution of  $\mathbb{Y}$  is

$$\begin{aligned} 0 \rightarrow R(-1, -5) \oplus R(-4, -3) \oplus R(-5, -2) \oplus R(-6, -1) \oplus R(-2, -4) \rightarrow \\ R(0, -5) \oplus R(-1, -4) \oplus R(-4, -2) \oplus R(-5, -1) \oplus R(-6, 0) \oplus R(-2, -3) \rightarrow R \rightarrow R/I_{\mathbb{Y}} \rightarrow 0. \end{aligned}$$

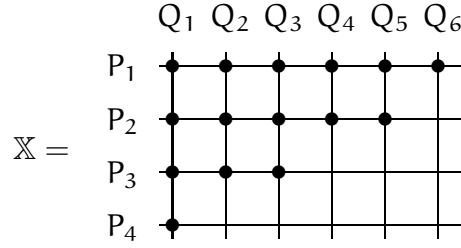
The resolution is not minimal in this case because although  $I_{\mathbb{Y}} = (I_{\mathbb{X}}, F)$ , one of the minimal generators of  $I_{\mathbb{X}}$  is actually a multiple of  $F$ . Precisely,  $G = L_1 L_2 L_3 R_1 R_2 R_3$  is a minimal generator of degree  $(3, 3)$  in  $I_{\mathbb{X}}$ , and clearly  $F = L_3 G$ .

#### 4. SEPARATORS OF ACM POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$ AND RESOLUTIONS

When  $\mathbb{X}$  is a set of ACM points in  $\mathbb{P}^1 \times \mathbb{P}^1$ , we can improve upon the results of the last section. We will show that the converse of Theorem 3.2 (i) holds for points in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Furthermore, we demonstrate that when  $\mathbb{X}$  is ACM, but  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  is not ACM, then the mapping cone construction used in proof of Theorem 3.2 gives a minimal resolution of  $I_{\mathbb{Y}}$ . In order to prove these results, we make use of properties of ACM sets of points in  $\mathbb{P}^1 \times \mathbb{P}^1$  developed in [12, 21, 22]; we begin with a review of this material.

**4.1. ACM sets of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ .** If  $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  is a finite set of points, let  $\pi_1(\mathbb{X}) = \{P_1, \dots, P_r\}$ , respectively,  $\pi_2(\mathbb{X}) = \{Q_1, \dots, Q_t\}$ , denote the distinct first coordinates, respectively, second coordinates, of the points  $\mathbb{X}$ . Each point in  $\mathbb{X}$  therefore can be written as  $P_i \times Q_j$  for some  $i$  and  $j$ ; the corresponding defining ideal is then  $I_{P_i \times Q_j} = (L_{P_i}, L_{Q_j}) \subseteq R = k[x_0, x_1, y_0, y_1]$  with  $\deg L_{P_i} = (1, 0)$  and  $\deg L_{Q_j} = (0, 1)$ .

We can associate to  $\mathbb{X}$  a tuple  $\lambda = (\lambda_1, \dots, \lambda_r)$  where  $\lambda_i = \#\{P \times Q \in \mathbb{X} \mid P = P_i\}$ . After relabeling the points, we can assume that  $\lambda_1 \geq \dots \geq \lambda_r$ . Note that  $\lambda$  is then a **partition** of  $|\mathbb{X}|$ . Associated to  $\lambda$  is another partition  $\lambda^* = (\lambda_1^*, \dots, \lambda_{\lambda_1}^*)$ , called the **conjugate** of  $\lambda$ , where  $\lambda_i^* = \#\{\lambda_j \in \lambda \mid \lambda_j \geq i\}$ . When  $\mathbb{X}$  is ACM, we can relabel the points so that  $\lambda_j^* = \#\{P \times Q \in \mathbb{X} \mid Q = Q_j\}$  (this can be deduced from [22, Theorem 4.8]). Thus, when  $\mathbb{X}$  is an ACM set of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ , by relabeling the points and permuting the lines of degree  $(1, 0)$  and  $(0, 1)$ , we can always assume that  $\mathbb{X}$  resembles the Ferrer's diagram of the partition  $\lambda$ . As an example, the set of points



is an ACM set of points corresponding to  $\lambda = (6, 5, 3, 1)$ . For this set of points  $\lambda^* = (4, 3, 3, 2, 2, 1)$ ; the first three in  $\lambda^*$  corresponds to the fact that there are three points which have second coordinate  $Q_2$ .

When  $\mathbb{X}$  is ACM, some of the algebraic invariants of  $I_{\mathbb{X}}$  can be deduced from  $\lambda$ .

**Theorem 4.1.** *Let  $\mathbb{X}$  be an ACM set of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ , and let  $\lambda = (\lambda_1, \dots, \lambda_r)$  be the associated partition.*

(i) *The minimal  $\mathbb{N}^2$ -graded resolution of  $R/I_{\mathbb{X}}$  has form*

$$0 \rightarrow \bigoplus_{(i,j) \in S_2} R(-i, -j) \rightarrow \bigoplus_{(i,j) \in S_1} R(-i, -j) \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0$$

where

$$\begin{aligned} S_1 &:= \{(r, 0), (0, \lambda_1)\} \cup \{(i-1, \lambda_i) \mid \lambda_i - \lambda_{i-1} < 0\}, \text{ and} \\ S_2 &:= \{(r, \lambda_r)\} \cup \{(i-1, \lambda_{i-1}) \mid \lambda_i - \lambda_{i-1} < 0\}. \end{aligned}$$

(ii) *Assume that the points of  $\mathbb{X}$  have been relabeled so that  $\mathbb{X}$  resembles the Ferrer's diagram of  $\lambda$ . Let  $\{i_1, \dots, i_l\} \subseteq \{1, \dots, r\}$  be the locations of the "drops" in  $\lambda$ , that is,*

$$\lambda_1 = \dots = \lambda_{i_1-1} > \lambda_{i_1} = \dots = \lambda_{i_2-1} > \lambda_{i_2} = \dots$$

*Then a minimal set of generators of  $I_{\mathbb{X}}$  is given by*

$$\{L_{P_1} \cdots L_{P_r}, L_{Q_1} \cdots L_{Q_{\lambda_1}}\} \cup \{G_1, \dots, G_l\}$$

*where  $G_k = L_{P_1} \cdots L_{P_{i_k-1}} L_{Q_1} \cdots L_{Q_{\lambda_{i_k}}}$  for  $k = 1, \dots, l$ .*

*Proof.* The statement (i) is [22, Theorem 5.1]. For (ii), because  $\mathbb{X}$  has been relabeled to resemble a Ferrer's diagram, it is straightforward to verify that each element of  $\{L_{P_1} \cdots L_{P_r}, L_{Q_1} \cdots L_{Q_{\lambda_1}}\} \cup \{G_1, \dots, G_t\}$  vanishes at all the points of  $\mathbb{X}$  and thus belongs to  $I_{\mathbb{X}}$ . To see that these are the minimal generators, it suffices to compare the degrees of each element with the elements in the set  $S_1$  from part (i).  $\square$

A set  $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  satisfies **property**  $(\star)$  if whenever  $P_1 \times Q_1$  and  $P_2 \times Q_2$  are two points in  $\mathbb{X}$  with  $P_1 \neq P_2$  and  $Q_1 \neq Q_2$ , then either  $P_1 \times Q_2 \in \mathbb{X}$  or  $P_2 \times Q_1 \in \mathbb{X}$  (or both) are in  $\mathbb{X}$ . We then have:

**Theorem 4.2** ([12, Theorem 4.3]). *A finite set of points  $\mathbb{X}$  in  $\mathbb{P}^1 \times \mathbb{P}^1$  is ACM if and only if  $\mathbb{X}$  satisfies property  $(\star)$ .*

**4.2. Separators and resolutions in  $\mathbb{P}^1 \times \mathbb{P}^1$ .** We begin by describing how to compute  $\deg_{\mathbb{X}}(P)$  for each point  $P \in \mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  when  $\mathbb{X}$  is ACM. A similar result was given by Marino [15, Proposition 7.4], but using the language of “left segments”.

**Lemma 4.3.** (i) *Let  $\{Q_1, \dots, Q_b\}$  be  $b \geq 2$  distinct points in  $\mathbb{P}^1$ , and let  $P_1$  be any point of  $\mathbb{P}^1$  (we allow the case that  $P_1 = Q_i$  for some  $i$ ). Consider the set of points*

$$\mathbb{X} = \{P_1 \times Q_1, P_1 \times Q_2, \dots, P_1 \times Q_b\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1.$$

*Then  $\mathbb{X}$  is ACM, and furthermore,  $\deg_{\mathbb{X}}(P_1 \times Q_1) = \{(0, b-1)\}$ .*

(ii) *Let  $\{P_1, \dots, P_a\}$  be  $a \geq 2$  distinct points in  $\mathbb{P}^1$ , and let  $Q_1$  be any point of  $\mathbb{P}^1$  (we allow the case that  $Q_1 = P_i$  for some  $i$ ). Consider the set of points*

$$\mathbb{X} = \{P_1 \times Q_1, P_2 \times Q_1, \dots, P_a \times Q_1\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1.$$

*Then  $\mathbb{X}$  is ACM, and furthermore,  $\deg_{\mathbb{X}}(P_1 \times Q_1) = \{(a-1, 0)\}$ .*

*Proof.* We only prove (i) since the second statement is similar. By Theorem 4.2, we have that  $\mathbb{X}$  is ACM, and so by Theorem 2.6, we have  $\deg_{\mathbb{X}}(P_1 \times Q_1) = \underline{\alpha}$  for some  $\underline{\alpha} \in \mathbb{N}^2$ . If  $L_{Q_i}$  denotes the degree  $(0, 1)$  form that passes through  $Q_i$ , then the form  $L_{Q_2} L_{Q_3} \cdots L_{Q_b}$  is a separator of degree  $(0, b-1)$ , so we must have  $(0, b-1) \succeq \underline{\alpha}$ .

If  $(0, b-1) \succ \underline{\alpha}$ , then there exists a separator  $F \neq 0$  with  $\deg F = \underline{\alpha} = (0, b')$  for some  $b' < b-1$ . On the other hand, the bigraded Hilbert function of  $\mathbb{Y} = \mathbb{X} \setminus \{P_1 \times Q_1\}$  is

$$H_{\mathbb{Y}} = \begin{bmatrix} 1 & 2 & 3 & \cdots & b-2 & b-1 & b-1 & \cdots \\ 1 & 2 & 3 & \cdots & b-2 & b-1 & b-1 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(we write  $H_{\mathbb{Y}}$  as an infinite matrix where the  $(i, j)$ th entry of the matrix equals  $H_{\mathbb{Y}}(i, j)$  where the indexing starts at 0) which implies that  $(I_{\mathbb{Y}})_{i,j} = 0$  for all  $(0, b-2) \succeq (i, j)$ . Since  $F \in (I_{\mathbb{Y}})_{(0, b')}$  with  $b' < b-1$  this means  $F = 0$ , a contradiction. So,  $\underline{\alpha} = (0, b-1)$ , as desired.  $\square$

When  $\mathbb{X}$  is an ACM set of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ , the degree of every point in  $\mathbb{X}$  is found by simply counting the points which share the same first and second coordinate.



**Theorem 4.4.** *Let  $\mathbb{X}$  be an ACM set of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ . For any  $P \times Q \in \mathbb{X}$  let*

$$\mathbb{X}_{P,1} = \{P \times Q, P \times Q_2, \dots, P \times Q_b\} \subseteq \mathbb{X}$$

*be all the points of  $\mathbb{X}$  whose first coordinate is  $P$ , and let*

$$\mathbb{X}_{Q,2} = \{P \times Q, P_2 \times Q, \dots, P_a \times Q\} \subseteq \mathbb{X}$$

*be all the points of  $\mathbb{X}$  whose second coordinate is  $Q$ . Then*

$$\deg_{\mathbb{X}}(P \times Q) = \{(|\mathbb{X}_{Q,2}| - 1, |\mathbb{X}_{P,1}| - 1)\} = \{(a - 1, b - 1)\}.$$

*Proof.* Because  $\mathbb{X}$  is ACM,  $\deg_{\mathbb{X}}(P \times Q) = \alpha$  for some  $\alpha \in \mathbb{N}^2$ . Let  $L_{P_i}$  be the degree  $(1, 0)$  form that passes through  $P_i$  for  $i = 2, \dots, a$  and let  $L_{Q_j}$  be the degree  $(0, 1)$  form that passes through  $Q_j$  for  $j = 2, \dots, b$ . We will show that  $F = L_{P_2} \cdots L_{P_a} L_{Q_2} \cdots L_{Q_b}$  is a minimal separator of  $P \times Q$ .

By construction,  $F(P \times Q) \neq 0$ . Now consider any point  $P' \times Q' \in \mathbb{X} \setminus \{P \times Q\}$ . If  $P' \in \{P_2, \dots, P_a\}$  or  $Q' \in \{Q_2, \dots, Q_b\}$ , then  $F(P' \times Q') = 0$ . So, suppose  $P' \notin \{P_2, \dots, P_a\}$  and  $Q' \notin \{Q_2, \dots, Q_b\}$ . Now  $\mathbb{X}$  satisfies property  $(\star)$  by Theorem 4.2. So, since  $P' \times Q'$  and  $P \times Q$  are in  $\mathbb{X}$ , then either  $P' \times Q \in \mathbb{X}$ , in which case  $P' \in \{P_2, \dots, P_a\}$ , a contradiction, or  $P \times Q' \in \mathbb{X}$ , in which case  $Q' \in \{Q_2, \dots, Q_b\}$ , a contradiction. Hence  $F$  is a separator for  $P$  of degree  $(a - 1, b - 1)$ , whence  $(a - 1, b - 1) \succeq \alpha$ .

Suppose that  $\alpha \prec (a - 1, b - 1)$ . So, there is a minimal separator  $F' \neq 0$  with  $\deg F' = \alpha = (\alpha_1, \alpha_2)$  with  $\alpha_1 < a - 1$  or  $\alpha_2 < b - 1$ . Suppose that  $\alpha_1 < a - 1$ . Now  $F'$  is also a separator for  $P \times Q$  from  $\mathbb{X}_{Q,2}$ . By Lemma 4.3,  $\deg_{\mathbb{X}_{Q,2}}(P \times Q) = (a - 1, 0)$ . So, we must have  $(a - 1, 0) \preceq \deg F'$ . But  $\alpha_1 < a - 1$ , which gives a contradiction. So,  $\alpha_1 \geq a - 1$ . A similar argument implies  $\alpha_2 \geq b - 1$ , and thus  $\deg_{\mathbb{X}}(P \times Q) = (a - 1, b - 1)$ .  $\square$

**Remark 4.5.** Suppose  $\lambda = (\lambda_1, \dots, \lambda_r)$  is the partition associated to  $\mathbb{X}$ , and  $\mathbb{X}$  resembles the Ferrer's diagram of  $\lambda$ . If  $P_i \times Q_j \in \mathbb{X}$ , then the conclusion of Theorem 4.4 is equivalent to  $\deg_{\mathbb{X}}(P_i \times Q_j) = (\lambda_i^* - 1, \lambda_i - 1)$ .

We now prove the converse of Theorem 3.2 for ACM points in  $\mathbb{P}^1 \times \mathbb{P}^1$ :

**Theorem 4.6.** *Let  $\mathbb{X}$  be an ACM set of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ , and suppose that*

$$0 \rightarrow \mathbb{F}_2 = \bigoplus_{(i,j) \in S_2} R(-i, -j) \rightarrow \mathbb{F}_1 \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0$$

*is the minimal  $\mathbb{N}^2$ -graded free resolution of  $R/I_{\mathbb{X}}$ . Let  $P \in \mathbb{X}$  be any point. Then  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  is ACM if and only if  $\deg_{\mathbb{X}}(P) + (1, 1) \in S_2$ .*

*Proof.* In light of Theorem 3.2 (i), it suffices to prove the converse statement. As noted above, we can assume that  $\mathbb{X}$  resembles a Ferrer's diagram of some partition  $\lambda$ . By Theorem 4.1 the shifts in  $\mathbb{F}_2$  are  $S_2 = \{(r, \lambda_r)\} \cup \{(i - 1, \lambda_{i-1}) \mid \lambda_i - \lambda_{i-1} < 0\}$ . We consider two cases: (1)  $\lambda = (\lambda_1, \dots, \lambda_1)$  and (2)  $\lambda = (\lambda_1, \dots, \lambda_r)$  with  $\lambda_1 > \lambda_r$ .

In the first case,  $S_2 = \{(r, \lambda_r)\} = \{(r, \lambda_1)\}$ . Moreover,  $\lambda = (\lambda_1, \dots, \lambda_1)$  if and only if  $\mathbb{X}$  is a complete intersection of type  $(\lambda_1, r)$ , that is,  $\mathbb{X}$  is a grid of  $\lambda_1 \times r$  points. By Lemma 4.4, each point  $P \in \mathbb{X}$  has  $\deg_{\mathbb{X}}(P) = (r - 1, \lambda_1 - 1)$ . So,  $\deg_{\mathbb{X}}(P) + (1, 1) \in S_2$ . But because  $\mathbb{X}$  is a complete intersection,  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  is ACM because  $\mathbb{Y}$  still satisfies property  $(\star)$ .

For the second case, suppose  $\deg_{\mathbb{X}}(P_i \times Q_j) + (1, 1) = (\lambda_j^*, \lambda_i) \in S_2$ . So, either  $(\lambda_j^*, \lambda_i) = (r, \lambda_r)$ , or there exists an  $i' > i$  such that  $(\lambda_j^*, \lambda_i) = (i' - 1, \lambda_{i'-1})$ . For the second statement, because  $\lambda_i \geq \lambda_{i+1} \geq \dots$  there exists some  $i' > i$  such that  $\lambda_i = \dots = \lambda_{i'-1} > \lambda_{i'}$ . Since  $(i' - 1, \lambda_{i'-1})$  is the only tuple in  $S_2$  whose second coordinate is  $\lambda_i = \lambda_{i'-1}$ , then  $(i' - 1, \lambda_{i'-1})$  is the element in  $S_2$  that is equal to  $\deg_{\mathbb{X}}(P_i \times Q_j) + (1, 1)$  and  $\lambda_j^* = i' - 1$ . So, if  $\lambda_j^* = r$ ,

$$\mathbb{X}_{Q_j,2} = \{P_1 \times Q_j, P_2 \times Q_j, \dots, P_i \times Q_j, \dots, P_r \times Q_j\}$$

is the set of all points in  $\mathbb{X}$  with second coordinate  $Q_j$ , and if  $\lambda_j^* = i' - 1$ , then

$$\mathbb{X}_{Q_j,2} = \{P_1 \times Q_j, P_2 \times Q_j, \dots, P_i \times Q_j, \dots, P_{i'-1} \times Q_j\} \subseteq \mathbb{X}$$

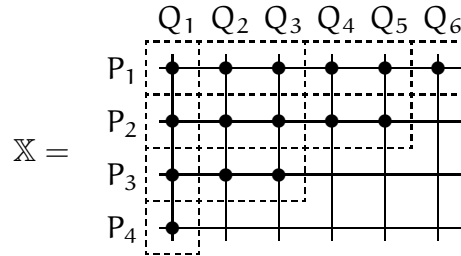
is the set of all the points in  $\mathbb{X}$  with second coordinate is  $Q_j$ .

Suppose, for a contradiction, that  $\mathbb{Y} = \mathbb{X} \setminus \{P_i \times Q_j\}$  is not ACM. Thus  $\mathbb{Y}$  does not satisfy property  $(\star)$ . We do the case that  $\lambda_i = \lambda_{i'-1}$  first. Because we have only removed the point  $P_i \times Q_j$ , this means that there exist points  $P_i \times Q'$  and  $P' \times Q_j$  in  $\mathbb{Y}$  with  $P' \times Q' \notin \mathbb{Y}$  (and clearly  $P_i \times Q_j \notin \mathbb{Y}$ ). Because  $\mathbb{X}$  has the shape of the Ferrer's diagram, we can take  $P' = P_c$  with  $c > i$ . To see this, note that the Ferrer's shape implies that if  $P_i \times Q' \in \mathbb{X}$ , then so are all the points  $P_k \times Q'$  with  $k < i$ . Thus, because  $P_i \times Q'$  is in  $\mathbb{X}$ , but  $P' \times Q' \notin \mathbb{X}$ , we must have  $P' = P_c$  with  $c > i$ .

On the other hand, again from the Ferrer's shape we must also have  $\lambda_i > \lambda_c$ , because  $P_i \times Q' \in \mathbb{X}$ , but  $P_c \times Q' \notin \mathbb{X}$ . But  $P_c \times Q_j \in \mathbb{X}_{Q_j,2}$ , and thus  $i < c \leq i' - 1$ . But then we have  $\lambda_i = \dots = \lambda_c = \dots = \lambda_{i'-1}$ , and thus  $\lambda_c = \lambda_i < \lambda_i$ , a contradiction. So,  $\mathbb{Y}$  must have property  $(\star)$ , and must be ACM by Theorem 4.2

In the case that  $\lambda_i = \lambda_r$ , a similar argument would show that there exists a point  $P_c \times Q' \in \mathbb{X}$  with  $c > r$  and  $\lambda_c < \lambda_r$ . But this is not possible since  $\lambda_r$  is the smallest entry of  $\lambda$ . So, again  $\mathbb{Y}$  must have property  $(\star)$ , and must be ACM.  $\square$

**Example 4.7.** We illustrate the above ideas with the following set of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ :



The associated partition is  $\lambda = (6, 5, 3, 1)$  and  $\lambda^* = (4, 3, 3, 2, 2, 1)$ .

We have divided the set of points into a series of boxes (the dashed boxes). Every point in the same box has the same degree. For example  $P_1 \times Q_2$  and  $P_1 \times Q_3$  both have degree  $(2, 5)$ . For this set of points, the shifts that appear at the end of the minimal resolution of  $R/I_{\mathbb{X}}$  are:

$$S_2 = \{(4, 1), (1, 6), (2, 5), (3, 3)\}.$$

The points in  $\mathbb{X}$  in the “outside” boxes, i.e., the box containing  $P_4 \times Q_1$ , the box containing  $P_3 \times Q_2$  and  $P_3 \times Q_3$ , the box containing  $P_2 \times Q_4$  and  $P_2 \times Q_5$ , and the box containing  $P_1 \times Q_6$ , all have the property that  $\deg_{\mathbb{X}}(P_i \times Q_j) + (1, 1) \in S_2$ . For example,

$\deg_{\mathbb{X}}(P_3 \times Q_3) + (1, 1) = (3, 3)$ . If we remove any point from these boxes, the resulting set of points will still be ACM. On the other hand, if we remove any point from an “inside” box, the resulting set of points will not be ACM. For example, if  $P_2 \times Q_3$  is removed, then  $\mathbb{X} \setminus \{P_2 \times Q_3\}$  no longer satisfies property  $(\star)$ , and thus is not ACM.

Let  $\nu(I)$  denote the minimal number of generators of a multihomogeneous ideal  $I$ .

**Lemma 4.8.** *Suppose that  $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  is ACM, but  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  is not ACM for some  $P \in \mathbb{X}$ . Then  $\nu(I_{\mathbb{Y}}) = \nu(I_{\mathbb{X}}) + 1$ .*

*Proof.* Because  $\mathbb{X}$  is ACM,  $|\deg_{\mathbb{X}}(P)| = 1$ . Let  $F$  be a minimal separator of  $\deg_{\mathbb{X}}(P)$ . By Theorem 2.5 we have  $I_{\mathbb{Y}} = (I_{\mathbb{X}}, F)$ , and hence  $\nu(I_{\mathbb{Y}}) \leq \nu(I_{\mathbb{X}}) + 1$ .

Let  $\lambda = (\lambda_1, \dots, \lambda_r)$  be the partition associated to  $\mathbb{X}$ , and relabel  $\mathbb{X}$  so  $\mathbb{X}$  resembles the Ferrer’s diagram of  $\lambda$ . Note that  $\lambda_1 > \lambda_r$ , because if  $\lambda_1 = \lambda_r$ , then  $\mathbb{X}$  would be a complete intersection, in which case  $\mathbb{X} \setminus \{P\}$  is ACM for all  $P \in \mathbb{X}$ .

Assume that  $P = P_i \times Q_j$ . Because  $\mathbb{Y} = \mathbb{X} \setminus \{P_i \times Q_j\}$  is not ACM, the set  $\mathbb{Y}$  does not satisfy  $(\star)$ . In particular, the points  $P_i \times Q_{\lambda_i}$  with  $\lambda_i > j$  and  $P_{\lambda_j^*} \times Q_j$  with  $\lambda_j^* > i$  are in both  $\mathbb{Y}$ , but neither  $P_i \times Q_j$  or  $P_{\lambda_j^*} \times Q_{\lambda_i}$  are in  $\mathbb{Y}$ . Note that this also implies that  $P_{\lambda_j^*} \times Q_{\lambda_i} \notin \mathbb{X}$ . By Theorem 4.4, we have  $\deg_{\mathbb{X}}(P_i \times Q_j) = (\lambda_j^* - 1, \lambda_i - 1)$ ; in particular, a minimal separator of this point is

$$F = L_{P_1} \cdots \hat{L}_{P_i} \cdots L_{P_{\lambda_j^*}} L_{Q_1} \cdots \hat{L}_{Q_j} \cdots L_{Q_{\lambda_i}}$$

where  $\hat{\phantom{x}}$  denotes the term is omitted.

By Theorem 4.1, if  $\{i_1, \dots, i_l\} \subseteq \{1, \dots, r\}$  are the locations of the “drops” in  $\lambda$ , then the minimal generators of  $I_{\mathbb{X}}$  are

$$\mathcal{M} = \{L_{P_1} \cdots L_{P_r}, L_{Q_1} \cdots L_{Q_{\lambda_1}}\} \cup \{G_1, \dots, G_l\}$$

where  $G_k = L_{P_1} \cdots L_{P_{i_k-1}} L_{Q_1} \cdots L_{Q_{\lambda_{i_k}}}$  for  $k = 1, \dots, l$ . If  $\nu(I_{\mathbb{Y}}) < \nu(I_{\mathbb{X}}) + 1$ , then because  $F \notin I_{\mathbb{X}}$ , there exists a minimal generator  $G$  such that

$$G = HF + \sum_{F_i \in \mathcal{M} \setminus \{G\}} H_i F_i.$$

Now, by degree considerations, if  $H_i \neq 0$ , then we must have  $\deg G \succeq \deg F_i$ . But by Theorem 4.1 (i), for any two minimal generators  $F_i, F_j$  of  $I_{\mathbb{X}}$ , we have  $\deg F_i \not\succeq \deg F_j$  and  $\deg F_j \not\succeq \deg F_i$ . Thus, if  $\nu(I_{\mathbb{Y}}) < \nu(I_{\mathbb{X}}) + 1$ , in the sum above we have  $H_i = 0$  for all  $i$ , and thus there must be a generator  $G$  such that  $G = HF$ , and hence  $\deg G \succeq \deg F$ .

Again, by degree considerations, since  $\deg F \succeq (1, 1)$ ,  $G \neq L_{P_1} \cdots L_{P_r}$  or  $L_{Q_1} \cdots L_{Q_{\lambda_1}}$ . So, consider  $i_k \in \{i_1, \dots, i_l\}$ . If  $i_k \leq i$ , then  $\deg G_k = (i_k - 1, \lambda_{i_k})$ . But then  $i_k \leq i < \lambda_j^*$ , and thus  $\deg G_k \not\succeq (\lambda_j^*, \lambda_i) = \deg F$ . On the other hand if  $i_k > \lambda_j^*$ , then  $\lambda_{i_k} < j < \lambda_i$ . But then  $\deg G_k = (i_k - 1, \lambda_{i_k}) \not\succeq (\lambda_j^* - 1, \lambda_i - 1) = \deg F$  since  $\lambda_{i_k} < \lambda_i - 1$ .

So, it remains to consider the case that  $i_k \in \{i_1, \dots, i_l\}$  and  $i < i_k \leq \lambda_j^*$ . We then have

$$G_k = L_{P_1} \cdots L_{P_{i_k-1}} L_{Q_1} \cdots L_{Q_{\lambda_{i_k}}} = H L_{P_1} \cdots \hat{L}_{P_i} \cdots L_{P_{\lambda_j^*}} L_{Q_1} \cdots \hat{L}_{Q_j} \cdots L_{Q_{\lambda_i}} = HF.$$

But since  $i < i_k$ , we have  $\lambda_i \geq \lambda_{i_k-1} > \lambda_{i_k}$ . Thus  $F$  cannot divide  $G_k$  since  $L_{Q_{\lambda_i}}$  divides  $F$ , but not  $G_k$ . We have thus shown that for every generator of  $I_{\mathbb{X}}$ ,  $F$  cannot divide it, thus providing the desired contradiction.  $\square$

By Theorem 3.2 (ii), if  $P \in \mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$  is chosen so that  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  is not ACM, then  $\text{depth}(R/I_{\mathbb{Y}}) = r - 1$ . When  $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ , we can prove a stronger result.

**Theorem 4.9.** *Let  $\mathbb{X}$  be an ACM set of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ , and suppose that*

$$0 \rightarrow \mathbb{F}_2 \rightarrow \mathbb{F}_1 \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0$$

*is the minimal  $\mathbb{N}^2$ -graded free resolution of  $R/I_{\mathbb{X}}$ . Let  $P \in \mathbb{X}$  be a point with  $\deg_{\mathbb{X}}(P) = (\alpha_1, \alpha_2)$ . Then  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  is not ACM if and only if it has a minimal  $\mathbb{N}^2$ -graded free resolution of type*

(4.1)

$$\begin{array}{ccccccc} & & \mathbb{F}_2 & & & & \\ & & \oplus & & & & \\ 0 \rightarrow R(-\alpha_1 - 1, -\alpha_2 - 1) \rightarrow & R(-\alpha_1 - 1, -\alpha_2) & \longrightarrow & \mathbb{F}_1 & \longrightarrow & R \rightarrow R/I_{\mathbb{Y}} \rightarrow 0 \\ & \oplus & & \oplus & & & \\ & R(-\alpha_1, -\alpha_2 - 1) & & R(-\alpha_1, -\alpha_2) & & & \end{array}$$

*Proof.* If  $\mathbb{Y}$  is a set of points in  $\mathbb{P}^1 \times \mathbb{P}^1$  with a minimal bigraded free resolution of type (4.1), since it has length 3, then  $\mathbb{Y}$  is not ACM. So, suppose that  $\mathbb{Y} = \mathbb{X} \setminus \{P\}$  is not ACM. As shown in the proof of Theorem 3.2,  $R/I_{\mathbb{Y}}$  has resolution of type (4.1). It suffices to show that the resolution is not minimal.

We first note that the resolution cannot be shortened since  $\text{depth}(R/I_{\mathbb{Y}}) < 2$ . Thus, if the resolution of (4.1) were not minimal, some shift in  $\mathbb{F}_1 \oplus R(-\alpha_1, -\alpha_2)$  would have to cancel out with some shift in  $\mathbb{F}_2 \oplus R(-\alpha_1 - 1, -\alpha_2) \oplus R(-\alpha_1, \alpha_2 - 1)$ . But if there were such a cancellation, that would imply that  $\nu(I_{\mathbb{Y}}) \leq \nu(I_{\mathbb{X}})$ , contradicting Lemma 4.8. Thus  $R/I_{\mathbb{Y}}$  has a minimal resolution of type (4.1).  $\square$

**Example 4.10.** Suppose we know  $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$  is not ACM, and in fact, we know the  $\mathbb{N}^r$ -graded resolution. It is tempting to speculate that the rank of the last syzygy module gives us information about the minimal number of points one should add to  $\mathbb{X}$  to make the set of points ACM. Unfortunately, no clear correspondence is known. For example, let  $P_i \in \mathbb{P}^1$  for  $i = 1, \dots, 5$  be distinct points and let  $\mathbb{X}$  be the following set of points of type  $P_{i,j} = P_i \times P_j$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ :

$$\mathbb{X} = \{P_{1,1}, P_{1,3}, P_{1,5}, P_{2,2}, P_{2,4}, P_{2,5}, P_{3,1}, P_{3,2}, P_{3,3}, P_{4,1}, P_{4,4}\}.$$

Then, using CoCoA, a resolution of  $R/I_{\mathbb{X}}$  has the form

$$0 \rightarrow R^4 \rightarrow R^{10} \rightarrow R^7 \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0$$

where we have suppressed all the bigraded shifts. The rank of the last syzygy module is 4. However, to make  $\mathbb{X}$  ACM, we need to add at least 5 points:  $P_{1,2}, P_{1,4}, P_{2,1}, P_{2,3}$ , and  $P_{3,4}$ .

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